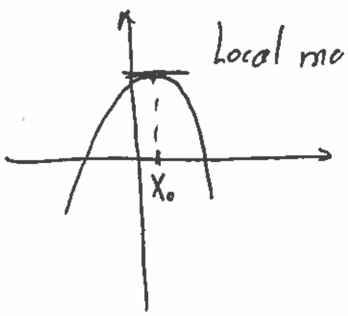
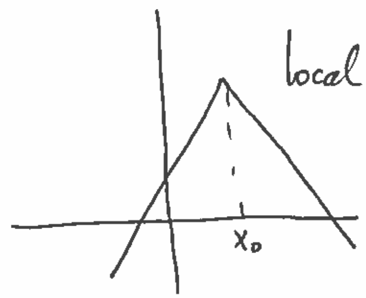
14.7 Maximum and minimum Values

In 1-D,





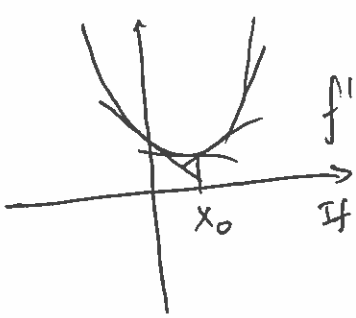
- Necessary condition for local maximum or local minimum when f is differentiable at x_0 is

$$f'(x_0) = 0.$$

- Sufficient condition. If f has 2nd derivative at x_0 and $f'(x_0) = 0$, then

  If $f''(x_0) > 0$, f has a local minimum at x_0

  If $f''(x_0) < 0$, f has a local maximum at x_0.



0.22 Definition 12: Local Maximum and Local Minimum

Let $\mathcal{D} \subseteq \mathbb{R}^2$ be the domain of the function f.

i) A function of two variables has a **local minimum at** (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) in some $\mathcal{B}_r(a, b)$ contained in $\mathcal{D}$. The number $f(a, b)$ is called a **a local minimum value of** f.

ii) A function of two variables has a **local maximum at** (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in some $\mathcal{B}_r(a, b)$ contained in $\mathcal{D}$. The number $f(a, b)$ is called **a local maximum value of** f.
Discuss the concepts of absolute maximum and minimum.

0.23 Theorem 7: Necessary Condition for a Relative Extrema

If the following two conditions are verified:
i) f has a local maximum or minimum at (a, b)
. and
ii)The first order partial derivatives $f_x(a, b)$ and $f_y(a, b)$ exist, then

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0$$

0.24 Definition 13: Critical Points

A point (a, b) is called a critical point of a function f if

$$f_x(a, b) = 0 \qquad \text{and} \qquad f_y(a, b) = 0,$$

or if one of these partial derivatives does not exist at (a, b).

0.25 Theorem 8: Test for Relative Extrema

Consider a function f defined on a domain $\mathcal{D}$ of $\mathbb{R}^2$, then if

i) There is a point (a, b) in $\mathcal{D}$, such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. It means the point (a, b) is a critical point of f.

ii) The second-order partial derivatives f_{xx}, f_{xy}, and f_{yy} are continuous in a disk $\mathcal{B}_r(a, b)$ contained in $\mathcal{D}$.

iii) $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - (f_{xy}(a, b))^2$,

then

a) If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f reaches a local minimum $f(a, b)$ at the critical point (a, b).

b) If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f reaches a local maximum $f(a, b)$ at the critical point (a, b).

c) If $D(a, b) < 0$, then $f(a, b)$ is not a relative extrema and there is a **saddle point on the graph of** f at the critical point (a, b).

d) If $D(a, b) = 0$ the test gives no information.

Why?
Consider $D_{\mathbf{u}}f$ in the direction of $\mathbf{u} = \langle h, k\rangle$, then $D_{\mathbf{u}}f = f_x h + f_y k$ and

$$D^2_{\mathbf{u}}f = f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2 = f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2 + \frac{k^2}{f_{xx}}(f_{xx}f_{yy} - f^2_{xy})$$

Example 1

$$z = f(x,y) = \frac{x^2}{4} - \frac{y^2}{9}$$

Differentiable everywhere.

$$f_x(x,y) = \frac{x}{2}, \qquad f_y(x,y) = -\frac{2}{9}y$$

Critical points:

$\frac{x}{2} = 0$ and $-\frac{2}{9}y = 0 \Rightarrow (0,0)$ critical point (unique)

Second Derivative test:

$$f_{xx}(x,y) = \frac{1}{2}, \qquad f_{yy}(x,y) = -\frac{2}{9}, \quad f_{xy}(x,y) = 0$$

$$\therefore \quad D(0,0) = f_{xx}(0,0)\, f_{yy}(0,0) - \left(f_{xy}(0,0)\right)^2$$

$$= \frac{1}{2}\left(-\frac{2}{9}\right) - 0 = -\frac{1}{9} < 0$$

Therefore,
f does not have a local max or local min at (a,b). The point (a,b) is a saddle point.

MAXIMA and minima

Example 3.-

Find all relative extrema for the function

$$z = f(x,y) = \frac{3}{2}y - \frac{1}{2}y^3 - x^2y + \frac{1}{16}$$

i) Critical Points:

$$f_x(x,y) = -2xy, \qquad f_y(x,y) = \frac{3}{2} - \frac{3}{2}y^2 - x^2$$

then C.P. should satisfy syst. of equations:

$$\begin{cases} -2xy = 0 \\ \frac{3}{2} - \frac{3}{2}y^2 - x^2 = 0 \end{cases}$$

then, a) $x = 0$ and $+\frac{3}{2}y^2 = \frac{3}{2} \Rightarrow y = \pm 1$

Two critical points: $(0,1)$ and $(0,-1)$.

b) $y = 0$ and $x^2 = \frac{3}{2} \Rightarrow \; = \pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{6}}{2}$

Two more critical points: $\left(\frac{\sqrt{6}}{2}, 0\right)$, and $\left(-\frac{\sqrt{6}}{2}, 0\right)$.

$$C.P_s = \left\{ (0,1), (0,-1), \left(\frac{\sqrt{6}}{2}, 0\right), \left(-\frac{\sqrt{6}}{2}, 0\right) \right\}.$$

ii) $f_{xx}(x,y) = -2y$, $\quad f_{yy}(x,y) = -3y$, $\quad f_{xy}(x,y) = -2x$

All exists and are conts. on $\mathbb{R}^2$.

$$\boxed{D = (-2y)(-3y) - (-2x)^2 = 6y^2 - 4x^2}$$

0

iii) Type of critical points:

At

a) $(0,1)$

$$D(0,1) = f_{xx}(0,1)\, f_{yy}(0,1) - [f_{xy}(0,1)]^2 =$$

$$= (-2)(-3) - 0 = 6 > 0$$

$$f_{xx}(0,1) = -2 < 0 \Rightarrow$$

at $(0,1)$ local max is reached

local max: $f(0,1) = \frac{3}{2} - \frac{1}{2} - 0 + \frac{1}{16} = \frac{17}{16}$.

b) At $(0,-1)$

$D(0,-1) = 6 > 0$ and $f_{xx}(0,-1) = 2 > 0$

then

local minimum: at $(0,-1)$ given by $f(0,-1) = -\frac{3}{2} + \frac{1}{2} + \frac{1}{16} = -\frac{15}{16}$

c) At $\left(\frac{\sqrt{6}}{2}, 0\right)$

$$D\left(\frac{\sqrt{6}}{2}, 0\right) = -4\,\frac{6}{4} = -6 < 0$$

At $\left(\frac{\sqrt{6}}{2}, 0\right)$ there is a saddle point.

d) At $\left(-\frac{\sqrt{6}}{2}, 0\right)$

$$D\left(-\frac{\sqrt{6}}{2}, 0\right) = -4\left(-\frac{\sqrt{6}}{2}\right)^2 = -4 \cdot \frac{6}{4} = -6 < 0$$

At $\left(-\frac{\sqrt{6}}{2}, 0\right)$ there is a Saddle point.